Ambiguity matters if you invest in many assets*

Yuki Shigeta†

Abstract

This study is concerned with practical performances of the multiple priors portfolio based on mean-variance preference. The multiple priors portfolio is designed to be robust to model uncertainty as known as ambiguity. The out-of-sample back test find the following two empirical properties of the multiple priors portfolio: 1) the multiple priors portfolio tends to have a better performance when the number of the assets is large, and 2) it has less turnovers than the standard mean-variance efficient portfolios.

Key words: Ambiguity Aversion, Model Uncertainty, Multiple Priors, Portfolio Selection

JEL Classification: D81, G11

The mean-variance efficient portfolio achieves in theory the most efficient balance between return on investment (measured by mean) and associated risk (measured by variance). However, many empirical studies report empirical failures of the mean-variance efficient portfolio (e.g., Jagannathan and Ma (2003) and DeMiguel et al. (2009)). One reason of these empirical failures is that estimation errors of the sample moments erode the theoretical advantage of the mean-variance efficient portfolio. On the other hand, researchers develop portfolio selection rules under inevitable estimation errors. A typical example is ambiguity aversion: a characteristic of preferences that people do not like the choice associated with the event whose occurrence probability is unknown. The portfolio selection rules based on ambiguity aversion are suggested by many scholars including Garlappi et al. (2007), Pflug et al. (2012) and Shigeta (2017). However, how these portfolios work in reality is less investigated. Therefore, this study examines performances of the portfolios in the back test.

Throughout the paper, I denote by μ and Σ the expected return vector and covariance matrix of the risky assets returns. Furthermore, I assume the number of the risky assets is N, and it is more than 2. For convenience, let $\mathbf{1}_N$ denote an N-dimensional vector whose all elements are 1, and I use I_N to denote an N-dimensional identity matrix. I also use the superscript $\top$ to indicate the transpose of a vector or matrix.

The multiple priors model is one typical example of decision making under ambiguity. Shigeta (2017) extends the multiple priors model suggested by Garlappi et al. (2007) to include the priors for variances and

*This is an extended abstract of the original paper.

†Graduate School of Economics, Kyoto university, Japan. E-mail address: sy46744@gmail.com

covariances, in which the investor solves the following max-min problem.

$$\begin{aligned} &\max_{w} \min_{(\theta,V)} (\mu+\theta)^\top w - \frac{\gamma}{2} w^\top (\Sigma + V) w, \\ &\text{subject to } \theta^\top (\Sigma+V)^{-1}\theta \le (\eta^\theta)^2, \\ &\qquad\qquad \|V\|_F^2 \le (\eta^V)^2 \|\Sigma\|_F^2, \\ &\qquad\qquad \text{and } w^\top \mathbf{1}_N = 1, \end{aligned} \tag{1}$$

where η^θ and η^V are positive constants, and $\|\ \cdot\ \|_F$ is the Frobenius norm operator. The problem (1) is a version of the max-min expected utility optimization developed by Gilboa and Schmeidler (1989). θ and V can be regarded as an estimation error of the mean vector μ and covariance matrix Σ. The investor first takes a minimum of his or her mean-variance objective over θ and V — so he or she considers the worst case in which the estimated moments deviate from true values, and next maximizes the objective over the portfolio w. The investor recognizes the errors by two conventional statistical measures: a credible interval of the mean vector and a relative least square errors of the covariance matrix. The two constants η^θ and η^V therefore represent the degrees of the investor's confidence that his or her estimation is true. Large η^θ and η^V imply that the investor is less confident in estimation accuracy. η^θ is the upper bound of the credible interval of the mean vector, so it represents the degree of confidence in the mean vector estimation. On the other hand, η^V represents the degree of confidence in the covariance matrix estimation. Garlappi et al. (2007) consider the case that there is only a mean estimation error, and Shigeta (2017) adds a variance estimation error.

According to Shigeta (2017), the optimal portfolio in the problem (1) is

$$w_{MP} := \frac{1}{\gamma + \eta^\theta/\psi^*} \left(\Sigma + \eta^V \|\Sigma\|_F I_N\right)^{-1} \left(\mu - \frac{B_{MP} - (\gamma + \eta^\theta/\psi^*)}{A_{MP}} \mathbf{1}_N\right), \tag{2}$$

where

$$\begin{aligned} A_{MP} &= \mathbf{1}_N^\top \left(\Sigma + \eta^V \|\Sigma\|_F I_N\right)^{-1} \mathbf{1}_N, \quad B_{MP} = \mu^\top \left(\Sigma + \eta^V \|\Sigma\|_F I_N\right)^{-1} \mathbf{1}_N, \\ C_{MP} &= \mu^\top \left(\Sigma + \eta^V \|\Sigma\|_F I_N\right)^{-1} \mu, \quad D_{MP} = A_{MP} C_{MP} - B_{MP}^2, \end{aligned}$$

and ψ^* is the unique positive solution to the following quartic equation.

$$A_{MP} (\psi^*)^2 = \frac{D_{MP}}{(\gamma + \eta^\theta/\psi^*)^2} + 1.$$

As shown in Shigeta (2017), the optimal portfolio w_{MP} has an analytical property when the parameters γ, η^θ, and η^V tend to go to infinity. If γ or η^θ tends to go to infinity, w_{MP} converges to the global minimum-variance portfolio under the multiple priors for covariances, i.e.,

$$\lim_{\gamma \text{ or } \eta^\theta \to \infty} w_{MP} = \frac{1}{A_{MP}} \left(\Sigma + \eta^V \|\Sigma\|_F I_N\right)^{-1} \mathbf{1}_N.$$

On the other hand, if η^V tends to go to infinity, w_{MP} converges to the equally weighted portfolio, i.e.,

$$\lim_{\eta^V \to \infty} w_{MP} = \frac{1}{N} \mathbf{1}_N.$$

Furthermore, the multiple priors optimal portfolio (2) does not satisfy the two-fund separation theorem. The two-fund separation theorem states that every mean-variance efficient portfolio is a linear combination of two given mean-variance efficient portfolios. However, for any pair of given portfolios, the multiple priors optimal portfolio (2) cannot be represented by a linear combination of the given portfolios for any η^V. Therefore, the multiple priors optimal portfolio (2) has different implication to portfolio selection in the standard mean-variance framework.

In the paper, I consider two multiple priors optimal portfolios. The first has static parameters η^θ and η^V, which is called MP. I set $\eta^\theta = 0.3$ and $\eta^V = 0.1$. The other has time-varying parameters η^θ and η^V. The dynamics of these parameters depends on the ad-hoc rule by Shigeta (2017) as follows. Let $\widehat{\mu}_t^M$ and $\widehat{\Sigma}_t^M$ be a sample mean vector and covariance matrix at time t based on past M observations. The time-varying η_t^θ and η_t^V are defined as

$$\eta_t^\theta = \sqrt{(\widehat{\mu}_t^M - \widehat{\mu}_t^L)^\top (\widehat{\Sigma}_t^M)^{-1} (\widehat{\mu}_t^M - \widehat{\mu}_t^L)}, \quad \text{and} \quad \eta_t^V = \alpha \frac{\|\widehat{\Sigma}_t^M - \widehat{\Sigma}_t^L\|_F}{\|\widehat{\Sigma}_t^M\|_F}, \tag{3}$$

where M and L are a length of estimation window with $M > L$, and α is a constant. The investor uses $\widehat{\mu}_t^M$ and $\widehat{\Sigma}_t^M$ as μ and Σ in the portfolio (2). Therefore, the ad hoc rule (3) implies that the investor's ambiguity for $\widehat{\mu}_t^M$ and $\widehat{\Sigma}_t^M$ is large if the recent sample-based estimators $\widehat{\mu}_t^L$ and $\widehat{\Sigma}_t^L$ deviate from the long-term sample-based estimators $\widehat{\mu}_t^M$ and $\widehat{\Sigma}_t^M$. Shigeta (2017) finds that the ad hoc rule works well in his brief back test. The value of α is 0.1 according to Shigeta (2017). This portfolio is called MP-TV in the paper.

I also consider the traditional mean-variance efficient portfolio. The traditional mean-variance efficient portfolio (dubbed MV) is a solution to the problem (1) when $\eta^\theta = \eta^V = 0$. Furthermore, the following three other portfolios are examined for benchmarks.[1] The first one is the global minimum-variance portfolio (dubbed GMV), which attains the smallest variance of all the portfolio containing only the risky assets. It is a solution to a minimization problem with the objective $w^\top \Sigma w$ subject to $\mathbf{1}_N^\top w = 1$. Analytically, GMV is expressed as $\Sigma^{-1}\mathbf{1}_N / \mathbf{1}_N^\top \Sigma^{-1} \mathbf{1}_N$. Next, the (normalized) tangency portfolio (dubbed TP) is expressed as $\Sigma^{-1}(\mu - r_f)/\mathbf{1}_N^\top \Sigma^{-1}(\mu - r_f)$ in which r_f is the risk-free rate. A unnormalized version of TP attains the highest Sharpe ratio in the presence of the risk-free asset. Finally, I consider the equally weighted portfolio (dubbed EW). EW invests the same amount of money in each asset. Its analytical expression is $\mathbf{1}_N/N$. In contrast to the other portfolios considered in the paper, EW does not have any information about distribution of the risky assets. Therefore, EW does not involve any estimation error while it also does not have any theoretical advantage.

In the paper, I conduct an out-of-sample back test in the absence of the risk-free asset. For the asset universe, five data sets of stock indexes returns are considered: ME/BM 6, ME/BM 25, Industry 10, Industry 49, and MSCI 4. ME/BM 6 consists of the six Fama-French style size- and book-to-market-sorted portfolios. Each of them is a value-wighted index of the stocks sorted by size and book-to-market. ME/BM 25 is a 25-groups version of ME/BM 6. ME/BM 6 and 25 are obtained from the Ken French website.[2] Industry 10 consists of returns on 10 industry portfolios in the United States (US). These industries are Consumer-Discretionary, Consumer-Staples, Manufacturing, Energy, High-Tech, Telecommunication, Wholesale and Retail, Health, Utilities, and Others. On the other hand, Industry 49 consists of returns on 49 industry port-

[1] I consider more portfolio selection rules in the original paper.

[2] I thank Ken French for providing the data set via his website.

folios in the US. Industry 10 and 49 are also obtained from the Ken French website. MSCI 4 consists of returns on the four MSCI dollar-valued international equity indexes including the US, the United Kingdom, Japan, and Germany. These MSCI indexes data are obtained from the Thompson Reuters Datastream. All the five data sets are monthly returns from January 1976 to December 2015. For the risk-free rate, I use monthly returns over the 90-day T-bill by the Ken French website. The investment horizon covers from January 1981 to December 2015. The length of estimation window is $M = 60$, and the length of MP-TV's recent estimation window L is 36.

The investor chooses his or her portfolio based on the available data at each time. At each time t, the investor computes the sample moments by the return data from $t - M$ to $t - 1$, and he or she invests in the portfolio aforementioned and based on the sample moments, so that he or she obtains investment returns depending on his or her portfolio. I compare performances of the portfolios by these returns.

Table 1: **Monthly Sharpe Ratios.** Estimation window is $M = 60$. The investment horizon is $T = 420$ from Jan 1981 to Dec 2015. γ is 5.

Portfolio	ME/BM 6	ME/BM 25	Industry 10	Industry 49	MSCI 4
Ambiguity-averse optimal portfolios					
MP	0.202	0.216	0.187	0.194	0.118
MP-TV	0.264	0.298	0.160	0.203	0.094
Standard mean-variance optimal portfolios					
MV	0.237	0.192	0.050	0.110	0.075
GMV	0.267	0.276	0.202	-0.003	0.129
TP	0.233	0.198	0.049	-0.002	-0.049
EW	0.150	0.153	0.167	0.143	0.120

Table 1 shows out-of-sample Sharpe ratios. The results shown in Table 1 are summarized in the following three points. First, the out-of-sample Sharpe ratios of MP-TV dominate those of MV and TP in all the data sets. The Sharpe ratio of MP-TV in Industry 10 is 0.160 while one of MV is 0.050, and the difference is significant at 0.1% level (the p-value is 0.999). However, the differences between MP-TV and MV in the other data sets are not significant at 5% level. As well as MP-TV, MP has a better performance.

Next, GMV also dominates MV excluding the data set Industry 49. The Sharpe ratio of GMV in Industry 10 is 0.160, and the difference is significant at 0.5% level (the p-value is 0.997). However, the Sharpe ratio of MV in Industry 49 is statistically significantly higher than that of GMV at 5% level (the p-value is 0.049). The differences between GMV and MV in the other data sets are not significant at 10% level. Furthermore, GMV also dominates TP except for Industry 49.

Comparing MP-TV with GMV, MP-TV has a tendency to dominate GMV when the number of the assets is large. In the ME/BM data sets, GMV defeats MP-TV in ME/BM 6 while MP-TV dominates GMV in ME/BM 25. Similarly, GMV has a larger out-of-sample Sharpe ratio in Industry 10 than MP-TV, but this relation becomes reversed in Industry 49: the out-of-sample Sharpe ratio of GMV is -0.003 while that of MP-TV is 0.203. In MSCI 4, GMV has a better performance than MP-TV. The differences between MP-TV and GMV are not significant at 10% level excluding Industry 49 (its p-value is 0.999). MP also dominates

GMV in Industry 49, but GMV has a better performance in the other data sets than MP. Considering how to design MP and MP-TV, MP and MP-TV take account of ambiguity for the asset returns, and the ambiguity becomes large when the number of the assets is large. Therefore, it is natural that MP and MP-TV have a better performance than GMV when the number of the asset is large, in other words, ambiguity is large.

Table 2: **Turnover.** The definition of the turnover is given in the equation (4). Estimation window is $M = 60$. The investment horizon is $T = 420$ from Jan 1981 to Dec 2015. γ is 5.

Portfolio	ME/BM 6	ME/BM 25	Industry 10	Industry 49	MSCI 4
Ambiguity-averse optimal portfolios					
MP	0.076	0.093	0.110	0.175	0.063
MP-TV	0.414	0.405	0.385	0.374	0.217
Standard mean-variance optimal portfolios					
MV	3.803	71.544	3.008	1099.435	0.491
GMV	0.417	1.900	0.310	4.218	0.087
TP	2.369	11.288	39.447	191.942	14.573
EW	0.015	0.017	0.023	0.032	0.025

MP and MP-TV also have an advantage in practice: their transaction is relatively small. Define the turnover of the portfolio w such that

$$\text{Turnover} = \frac{1}{T-M}\sum_{t=M+1}^{T}\sum_{j=1}^{N}|w_{j,t}^{+} - w_{j,t}|, \tag{4}$$

where $w_{j,t}$ is the portfolio w's weight of the asset j at time t, and $w_{j,t}^{+}$ is the *ex post* portfolio w's weight of the asset j at time t. If the turnover of some portfolio is large, it is difficult to implement this portfolio because the transaction due to the turnover erodes the returns. Table 2 shows out-of-sample turnovers. For any data set, EW's turnover is the smallest. In Industry 49 data set, the turnover of EW is 0.032, and ones in the other data sets are less than 0.1. Since EW does not vary depending on estimation, this result is natural. On the other hand, MV and TP have a large transaction in all the datasets. In relatively larger data sets, ME/BM 25 and Industry 49, the turnovers of MV and TP are more than 10. The turnovers of GMV are also large in the large data sets, which are over 1, but those in small data sets are relatively small. However, the turnovers of MP and MP-TV are small relative to MV, GMV, and TP. These turnovers in every dataset are smaller than 0.5. By the definition, MP and MP-TV can be regard as a nonlinear combination between MV and EW, so it is plausible that their turnovers become small.

Table 3 shows the out-of-sample Sharpe ratios adjusted to transaction. I set the transaction cost as 50 basis points, so, for example, the investor needs to pay 5 dollars at time t when the transaction $\sum_{j=1}^{N}|w_{j,t}^{+} - w_{j,t}|$ is 1000 dollars. Taking account of transaction, MV and TP become largely worse. MV has negative Sharpe ratios in ME/BM 25, Industry 10 and Industry 49 while these Sharpe ratios are positive without transaction. As seen in Table 2, MV and TP have large turnovers, and their negative effects to portfolio performances are also large. Furthermore, the negative impacts of the turnovers are also large for the out-of-sample Sharpe ratio of GMV in ME/BM 25 and Industry 49. In ME/BM 6, GMV has a larger Sharpe

Table 3: **Monthly Sharpe Ratios with Transactions.** The transaction cost is 50 basis points. Estimation window is $M = 60$. The investment horizon is $T = 420$ from Jan 1981 to Dec 2015. γ is 5.

Portfolio	ME/BM 6	ME/BM 25	Industry 10	Industry 49	MSCI 4
Ambiguity-averse optimal portfolios					
MP	0.195	0.207	0.174	0.176	0.112
MP-TV	0.217	0.247	0.117	0.160	0.069
Standard mean-variance optimal portfolios					
MV	0.056	-0.245	-0.069	-0.186	0.034
GMV	0.215	0.052	0.156	-0.328	0.121
TP	0.064	-0.114	-0.049	0.044	-0.078
EW	0.150	0.153	0.166	0.141	0.120

ratio than MP in the absence of transaction, but the Sharpe ratio of MP with transaction becomes larger than that of GMV. On the other hand, the negative impacts of the turnovers to MP and MP-TV are limited even if the number of the assets is large. In ME/BM 25 and Industry 49, MP and MP-TV have a better performance than GMV, and the differences of their Sharpe ratios from the GMV's Sharpe ratio are significant at 0.1% level (the p-values are more than 0.999). Therefore, the multiple priors portfolios, MP and MP-TV are robust to transaction, and that is an advantage when implementing these portfolios in reality.

References

DeMiguel, V., Garlappi, L., Uppal, R., 2009. Optimal Versus Naive Diversification: How Inefficient Is the $1/N$ Portfolio Strategy? Review of Financial Studies 22, 1915–1953.

Garlappi, L., Uppal, R., Wang, T., 2007. Portfolio Selection with Parameter and Model Uncertainty: A Multi-Prior Approach. Review of Financial Studies 20, 41–81.

Gilboa, I., Schmeidler, D., 1989. Maxmin Expected Utility with Non-Unique Prior. Journal of Mathematical Economics 18, 141–153.

Jagannathan, R., Ma, T., 2003. Risk Reduction in Large Portfolios: Why Imposing the Wrong Constraints Helps. The Journal of Finance 58, 1651–1683.

Pflug, G. C., Pichler, A., Wozabal, D., 2012. The $1/N$ Investment Strategy Is Optimal under High Model Ambiguity. Journal of Banking and Finance 36, 410–417.

Shigeta, Y., 2017. Portfolio Selections under Mean-Variance Preference with Multiple Priors for Means and Variances. Annals of Finance 13, 97–124.