Equilibrium Selection under Selection-Mutation Dynamics among Neutrally Stable Strategies in Games of Language Evolution

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Abstract

This note argues signaling games when the number of states of the world is not the same as the number of signals. We investigate the stability of perturbed rest points of signaling games under the selection-mutation dynamics. We show that there exist perturbed rest points close to the particular form of neutrally stable strategies of signaling games that are asymptotically stable if we choose mutation rates of a sender and a receiver properly. These strategies are signaling systems. Perturbed rest points close to signaling systems satisfy asymptotic stability in the both cases where the number of states of the world is the same as the number of signals and not.

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1. Introduction

The signaling model of game theory is introduced by David Lewis (1969) in order to explain the theory of the meaning in natural language. In signaling games, there are a sender and a receiver. A sender is informed about states of the world but doesn't act for the information directly.

A receiver is not informed about states of the world but takes an action for signals that a sender sends. A sender has access to a set of arbitrary signals and sends a signal of them to a receiver. A receiver has a set of arbitrary actions. A receiver gets a signal from a sender and takes an action of them. If a receiver takes a proper action for the state of the world, both a sender and a receiver get a payoff of 1 and 0, otherwise.

Perfect communications occur when all one to one correspondences between states, signals and actions occur perfectly. Perfect communications are evolutionarily stable strategies only in the case where the number of states of the world is equal to the number of the signals. In the case where the number of states is not equal to the number of the signals, perfect communications are neutrally stable strategies. Perfect communications in the both cases are called signaling systems (David Lewis, 1969).

In this note, we focus on signaling systems under the selection-mutation dynamics in the case where the number of states is not equal to the number of the signals.

Under the replicator dynamics, strategies that satisfy asymptotic stability are only evolutionarily stable strategies, whereas the neutral strategy satisfies Lyapunov stability (Huttegger, 2007; Pawlowitsch, 2008). Thus, Hofbauer and Hutteger (2008, 2015) introduced the selection-mutation dynamics to investigate whether the perturbations of replicator dynamics led to the qualitative change in the dynamic behavior.

The selection-mutation dynamics coincides with the replicator dynamics when mutation rates equal zero.

Hofbauer and Hutteger (2008) analyzed binary signaling games and showed that perturbed rest points close to neutrally stable strategies satisfy asymptotic stability with the proper mutation rates of senders and receivers.

In signaling games of three states and three signals, there is no strategy but evolutionarily stable strategy that is asymptotically stable (Hofbauer and Hutteger, 2015). Hofbauer and Hutteger conjectured that in signaling games of four states and four signals, there is no strategy but evolutionarily stable strategy that is asymptotically stable.

Uchida, Miyashita and Fukuzumi (2016) analyzed the cases where the number of the states are not equal to signals. Uchida, Miyashita and Fukuzumi showed that in signaling games of two states and three signals, three states and four signals, there are neutrally stable strategies that are asymptotically stable. All these strategies are signaling systems.

In this note, we show that signaling systems generally satisfy asymptotic stability in the both cases where the number of states of the world is equal to the number of the signals and not.

2. Model

A signaling game has a sender and a receiver. Suppose that there are n states of the world and m signals. In this note, we analyze the case $n<m$ where there are more signals than states of the world.

A sender's pure strategy is represented by an n × m matrix $P \in P_{n\times m}$. If $p_{ij} = 1$, a sender sends a signal j given that a state i has occured. Otherwise, $p_{ik} = 0$, for all k, $k\neq j$.

A receiver's pure strategy is represented by an m × n matrix $Q_{m\times n}$. If $q_{ji}=1$, a receiver associates a state i with a signal j. Otherwise, $q_{jk}=0$ for all k, $k\neq i$.

Next, we introduce a mix strategy of a sender and a receiver. A mixed sender strategy is given by a stochastic n × m matrix $P \in P^{\Delta}_{n\times m}$ with $p_{ij} \geq 0$ for all $i=1, 2, \ldots, n$, $j=1, 2, \ldots, m$ and $\sum_{j=1}^{m} p_{ij}=1$ for all $i=1, 2, \ldots, n$.

Similarly, a mixed receiver strategy is given by a stochastic m × n matrix $Q \in Q^{\Delta}_{m\times n}$ with $q_{ji} \geq 0$ for all $j=1, 2, \ldots, m$, $i=1, 2, \ldots, n$ and $\sum_{i=1}^{n} q_{ji}=1$ for all $j=1, 2, \ldots, m$.

The payoff function of both a sender and a receiver in this game is given by

$$\pi(P,Q) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}q_{ji} = \frac{1}{n}tr(P,Q), \qquad (1)$$

Pawlowitsch(2008) gives a characterization of the best-response properties in this game. B denotes the best response correspondence.

Lemma 1. Let $B(Q) \in P$ and $B(P) \in Q$ denote each sets of each best responses.

(i) if $Q \in B(P)$, then

$\sum_{i\in argmax_i(\bar{p}_{ij^*})} q_{j^*i} = 1$ and $q^{j^*i}=0 \quad \forall\, i \notin argmax_i(\bar{p}_{ij^*})$;

(ii) if $P \in B(Q)$, then

$\sum_{j \in argmax_j(\bar{q}_{ji^*})} p_{i^*j} = 1$ and $p^{i^*j} = 0$ $\forall i \notin argmax_j(\bar{q}_{ji^*})$.

A strategy that is a best response to itself is a Nash strategy. In this game, there is an abundance of Nash equilibrium. We introduce some concepts of equilibrium.

Definition 1. A strategy *(P, Q)* is evolutionarily stable if
(i) it is a Nash strategy, and if
(ii) whenever for $(P', Q') \in P^{\Delta}_{n\times m} \times Q^{\Delta}_{m\times n} \setminus \{(P,Q)\}$, $\pi(P,Q) > \pi(P',Q')$.

A strict Nash equilibrium is evolutionarily stable and exists only in the case where *n*=*m* (Trapa and Nowak, 2000). Pawlowitsch(2008) introduced a weaker notion of stability.

Definition 2. A strategy *(P, Q)* is neutrally stable if
(i) *(P, Q)* is a Nash strategy, and if
(ii) whenever for $(P', Q') \in P^{\Delta}_{n\times m} \times Q^{\Delta}_{m\times n} \setminus \{(P,Q)\}$, $\pi(P,Q) \geq \pi(P',Q')$.

In this note, we focus on signaling systems. Signaling systems are evolutionarily stable strategies in the case *n*=*m*, whereas those are neutrally stable strategies in the case *n*≠*m.*

Definition 3. The Strategies *(P, Q)* are signaling systems in the case *n*≠*m* if and only if for all *i, j,* with $1 \leq i, j \leq min\{m, n\}$, $p_{ij}=q_{ji}=1$ or $p_{ij}=q_{ji}=0$.

In the signaling games with *n* states of the world and *m* signals, there are $\frac{n!}{(n-m)!}$ signaling systems.

In this note, we investigate the dynamic behavior of signaling games under the selection-mutation dynamics.
The selection-mutation dynamics for two populations is given by

$$\dot{x} = x_i((Ay)_i - \dot{\mathbf{x}}Ay) + \varepsilon(1 - mx_i), \qquad (2)$$

$$\dot{y} = y_j((Bx)_j - \dot{\mathbf{y}}Bx) + \delta(1 - ny_j), \qquad (3)$$

where x $\in S_m$ and $\mathbf{y} \in S_n$ are the state of each population, *(A, B)* are the payoff matrices and $\varepsilon,\ \delta$ are small, uniform mutation parameters.

If ε=δ=*0*, the selection-mutation dynamics coincides with the two population replicator dynamics.

3. Result

We investigate the selection-mutation dynamics close to signaling systems. First, we show that there exists a unique family of perturbed rest points that converges to the signaling systems of neutral stable strategies whenever $\varepsilon,\ \delta$ are sufficiently small. Next, we calculate the eigenvalue of the Jacobian Matrics of the selection-mutation dynamics and show that the Jacobian matrix has all negative eigenvalues.

Theorem 1. There exists a unique family of perturbed rest points that converge to the signaling system

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$$\mathrm{P} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \cdots & 0 & \vdots & \cdots & \vdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \ddots & 0 & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \end{pmatrix},\ \mathrm{Q} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \cdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \cdots & 1 \\ q_{n+1,1} & \cdots & q_{n+1,i} & \cdots & q_{n+1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ q_{m,1} & \cdots & q_{m,i} & \cdots & q_{m,n} \end{pmatrix}$$

where for all *i, j* with $1 \leq i \leq n,\ n+1 \leq j \leq m,\ 0 \leq q_{ji} \leq 1$ and $\sum_{i=1}^{n} q_{ji} = 1$, as ε, $\delta \to 0$. To a first approximation, these perturbed rest points are given by

$$\widetilde{\mathrm{P}} = \begin{pmatrix} 1+\frac{2m-mn-1}{n-1}\varepsilon & \varepsilon & \varepsilon & \cdots & \varepsilon & \frac{n}{n-1}\varepsilon & \cdots & \frac{n}{n-1}\varepsilon \\ \varepsilon & \ddots & \varepsilon & \cdots & \varepsilon & \vdots & \cdots & \vdots \\ \varepsilon & \varepsilon & 1+\frac{2m-mn-1}{n-1}\varepsilon & \varepsilon & \varepsilon & \frac{n}{n-1}\varepsilon & \cdots & \frac{n}{n-1}\varepsilon \\ \varepsilon & \cdots & \varepsilon & \ddots & \varepsilon & \vdots & \cdots & \vdots \\ \varepsilon & \cdots & \varepsilon & \varepsilon & 1+\frac{2m-mn-1}{n-1}\varepsilon & \frac{n}{n-1}\varepsilon & \cdots & \frac{n}{n-1}\varepsilon \end{pmatrix},$$

$$\tilde{Q} = \begin{pmatrix} 1-(n-1)\delta & \delta & \delta & \cdots & \delta \\ \delta & \ddots & \delta & \cdots & \delta \\ \delta & \delta & 1-(n-1)\delta & \delta & \delta \\ \delta & \cdots & \delta & \ddots & \delta \\ \delta & \cdots & \delta & \cdots & 1-(n-1)\delta \\ \frac{1}{n} & \cdots & \frac{1}{n} & \cdots & \frac{1}{n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{n} & \cdots & \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}.$$

Theorem 2. Let $(\tilde{P},\tilde{Q})$ be the perturbed rest point given by Theorem 1. Then, the following statement is true for sufficient small mutation rates:

If $-\frac{mn-n^2-1}{n-1}\varepsilon + n\delta > 0$, the Jacobian matrix of the selection-mutation dynamics has all negative eigenvalues.

4. Conclusion

In this note, we investigate the stability of perturbed rest points of signaling games under the selection-mutation dynamics in the case $n<m$.

We particularly focus on signaling systems that are neutrally stable strategies. We show that perturbed rest points close to signaling systems are asymptotically stable. Signaling systems are neutrally stable in the case $n>m$ whereas signaling systems are evolutionarily stable strategies in the case $n = m$. But, the asymptotic stability of signaling systems in the case $n = m$ keeps even in the case $n<m$.

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