An Axiomatic Model of Reference Dependence under Uncertainty

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Abstract

This paper presents a behavioral characteization of a reference-dependent choice under uncertainty in the Anscombe-Aumann framework. In this model, by using a reference prior, an act is evaluated by the weighted sum of the following two terms: the expected value with the reference prior and the difference between the evaluation of the act for each state and the expected value. The evaluation of the act with the reference prior has a role of the reference level of acts. This paper gives a possible explanation for an example which is not consistent with multiple priors models.

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1. Introduction

1.1. Motivation

Behavioral Economics has been developing since the seminal work of Kahneman and Tversky(1979), Prospect Theory, proposed. Gilboa(2009) states that the most important idea in Prospect Theory is the notion of *gain-loss asymmetry*. This means that when people are faced with the prospect of gaining something they do not have, they make decisions in certain way. This reaction is consistent with the assumption of risk aversion. However, when people are faced with the prospect of losing something that they already have, people react differently. Kahneman and Tversky(1979) argues that such reaction is due to loss aversion, which implies that people dislike losing something they have extremely. The source of such behaviors stems from one of the features of the cognitive mechanism. Actually, people react to changes, and not to absolute levels.

The goal of this paper is to incorporate this feature of human behaviors into the standard decision model under uncertainty, that is, the Anscombe-Aumann framework. The proposed model points out that the decision maker evaluates an act with a reference prior. By using the reference prior, the decision maker calculates the expected value of the act. The value is regarded as the reference level of acts. Then, with the expected value, the decision maker pays attention to the difference between the the evaluation of the act and the expected value for each state. Formally, an act f is evaluated as the following functional:

$$v(f)=\theta\sum_{s\in S}u(f(s))q(s)+(1-\theta)\sum_{s\in S}\left[u(f(s))-\sum_{s\in S}u(f(s))q(s)\right], \tag{1}$$

where u: $X\rightarrow\mathbb{R}$ is a non-constant affine function, q is a reference prior on S, θ captures the relative weight between the expected value of an act with a reference prior and the difference between the evaluation of the act and the expected value of the act for each state.

The rest of the paper is organized as follows. In subsection 1.2, we provide a brief literature review. In Section 2, we present a behavioral characterization of our model. In Section 3, we discuss our model with Machina(2009)'s reflection example.

1.2. Related Literature

We notice that this paper is related to the following branches of decision analysis: reference dependent choices and decisions under uncertainty or ambiguity. We briefly explain some of literature which is related to our analysis.

In the literature of reference-dependent choice models, Kozsegi and Rabin(2006) studies the model of reference-dependent choice. The reference point relies on expectations, and is characterized by equilibrium conditions. Ok et al. (2015) also studies a reference-dependent choice model where the reference point is determined endogenously. The reference point is captured by a choice problems, i.e., menu effects.

In the literature of decisions under uncertainty, let us notice the following literature. Gilboa and Schmeidler(1989)'s maxmin expected utility is the seminal work of this field. In this model, the decision maker has a set of subjective beliefs, and evaluates an act with the worst belief in the set of her subjective beliefs. Maccheroni et al. (2006) extends this framework to capture the several attitude toward uncertainty or ambiguity. Siniscalchi (2009)'s vector expected utility model has a baseline prior, which is similar to a reference prior of our model. The difference between the vector expected utility model and our model is that Siniscalchi (2009) adjusts the decision maker's perception of uncertainty or ambiguity by using the baseline prior. On the other hand, in our model, the decision maker pays attention to the difference between the evaluation of acts for each state and the expected value with the reference prior.

2. Model and Results

Let us introduce some notation. Consider a set S of *states of the world*, a sigma-algebra Σ of subsets of S called *events*, and a set X of *consequences*. Let $\mathcal{F}$ denote the set of all acts. Acts are denoted by f, a finite-valued function f: $S \rightarrow X$, which is Σ-measurable. In other words, an act is a Σ-measurable function from S to X. Let us detonate a constant act by x assigning the consequence x for each state $s \in S$. $B_0(\Sigma)$ is the set of all real-valued Σ-measurable simple functions. Note that $u(f) \in B_0(\Sigma)$ where u: $X \rightarrow \mathbb{R}$.

The primitive of the model is a binary relation $\succsim$ on $\mathcal{F}$. The asymmetric and symmetric parts are denoted by $\succ$ and $\sim$, respectively. If $f \in \mathcal{F}$, an element $x_f \in X$ is a *certainty equivalent* for f if $f \sim x_f$.

We assume that X is a convex subset of a vector space. We define convex combinations in a usual manner. Mixtures of acts are taken as follows: for any $f, g \in \mathcal{F}$ and $\alpha \in [0,1]$, an act $\alpha f +$ $(1\text{-}\alpha)g \in \mathcal{F}$ yields $\alpha f(s) + (1\text{-}\alpha)g(s) \in X$ for any $s \in S$.

2.1. Axioms

Axioms 1-3 are standard.

Axiom 1(Weak Order): $\succsim$ is complete and transitive.

Axiom 2(Continuity): For any $f, g, h \in \mathcal{F}$, the sets $\{\alpha \in [0,1] \mid \alpha f + (1\text{-}\alpha)g \succ h\}$ and $\{\alpha \in [0,1] \mid h \succ \alpha f + (1\text{-}\alpha)g\}$ are closed.

Axiom 3(Monotonicity): For any $f, g \in \mathcal{F}$, $f(s) \succsim g(s)$ for any $s \in S$ implies $f \succsim g$. Moreover, if $f(s) \succ g(s)$ for any $s \in S$, then $f \succ g$.

Next, we weaken the independence axiom introduced in Anscombe and Aumann(1963) as well as Gilboa and Schmeidler(1989). This axiom requires that the independence axiom restricts with only constant acts.

Axiom 4(Certainty Independence): For any $f, g \in \mathcal{F}$, $x \in X$, and $\alpha \in [0,1]$, $f \succsim g$ if and only if $\alpha f + (1\text{-}\alpha)x \succsim \alpha g + (1\text{-}\alpha)x$.

Axiom 5(Non-Degeneracy): For some $f, g \in \mathcal{F}$, $f \succ g$.

Lastly, we introduce the key axiom in our model. Before stating the axiom, let us introduce the following definition of pairs of acts. This definition is introduced by Siniscalchi (2009).

Definition 1: Two acts $f, \bar{f} \in \mathcal{F}$ are *complementary* if and only if, for any two states $s, s' \in S$,

$$\frac{1}{2}f(s)+\frac{1}{2}\bar{f}(s) \sim \frac{1}{2}f(s')+\frac{1}{2}\bar{f}(s')$$

We say that the pair $(f, \bar{f})$ is a *complementary pair*, if two acts $f, \bar{f} \in \mathcal{F}$ are *complementary.* Intuitively speaking, it can be seen that the utility profiles of pairs of acts are "mirror images." The intuition is that, under Axiom1 - 5, the preference relation $\succsim$ is represented by u, and two complementary acts are evaluated by $u \circ f$ and $u \circ \bar{f}$, which satisfies $u \circ \bar{f} = c - u \circ f$ for some real number $c \in \mathbb{R}$.

Axiom 6(Dominance): For any complementary pairs $(f, \bar{f})$ and $(g, \bar{g})$ in $\mathcal{F}$, and for any $\alpha \in (0,1)$, if the following two conditions are satisfied :

(i) $\bar{f} \succsim \bar{g}$,

(ii) $\alpha f + (1\text{-}\alpha)\bar{f} \succsim \alpha g + (1\text{-}\alpha)\bar{g}$.

then, $f \succsim g$.

This axiom says that if the mirror acts of f and g has the relationship such that $\bar{f}$ is weakly preferred to $\bar{g}$, and the mixture act $\alpha f + (1\text{-}\alpha)\bar{f}$ is weakly preferred to $\alpha g + (1\text{-}\alpha)\bar{g}$, then f is weakly preferred to g. This cognitive setting requires that the ranking between f and g is consistent with the mirror image of f and g, and the mixtures of two complementary pairs $(f, \bar{f})$

and $(g, \bar{g})$.

2.2. Representation Theorem

Theorem 1: *The following statements are equivalent:*

(a) $\succsim$ satisfies Weak Order, Continuity, Monotonicity, Certainty Independence, and Dominance.

(b) There exists a triple (u, θ, q) where $u: X \to \mathbb{R}$ is a non-constant affine function, $\theta \in [0,1]$, and q is a probability distribution over S such that for any $f, g \in \mathcal{F}$,

$$f \succsim g \Leftrightarrow \theta \sum_{s\in S} u(f(s))q(s) + (1-\theta)\sum_{s\in S}\left[u(f(s)) - \sum_{s\in S} u(f(s))q(s)\right] \geq \theta \sum_{s\in S} u(g(s))q(s) + (1-\theta)\sum_{s\in S}\left[u(g(s)) - \sum_{s\in S} u(g(s))q(s)\right]$$

Moreover, if two pairs (u, θ, q) and (u', θ', q') represent the same preference $\succsim$, then there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $u = \alpha u' + \beta$, $\theta = \theta'$, and $q = q'$.

In Theorem 1, the key parameter θ captures the relative weight between the expected value with the reference prior and the difference between the evaluation of acts and the expected value for each state. If $\theta = 1$, then the decision maker evaluates an act with a reference prior. This behavior is regarded as a standard subjective expected utility model. If $\theta = 0$, then the reference prior is used as the role of calculating the expected value. The decision maker focuses only on the difference from the reference level for each state.

2.3. Sketch of proof

We provide a sketch of proof of sufficiency in Theorem 1. We have the following three steps. First, by the argument of Maccheroni et al. (2006), we show that there exist a non-constant affine function (von-Neumann Morgenstein function) u and a normalized functional I which represents $\succsim$ on $\mathcal{F}$ satisfying the axioms (Axiom 1-5) in Theorem 1. Next, we show that our key axiom, *Dominance*, is represented by a functional I. Finally, we characterize the representation.

Lemma 1: *$\succsim$ on $\mathcal{F}$ satisfies Weak Order, Continuity, Monotonicity, Certainty Independence, and Non-Degeneracy if and only if there exist a non-constant affine function $u: X \to \mathbb{R}$ and a normalized functional $I: B(\Sigma,u(X)) \to \mathbb{R}$ such that $f \succsim g \Leftrightarrow I(u \circ f) \geq I(u \circ g)$.*

The formal proof is shown in Maccheroni et al. (2006). Furthermore, the proof of the uniqueness result is straightfoward. We omit the proof.

Remark: *If two pairs (u, I) and (u', I') represent the same $\succsim$, then there exist $\alpha > 0$, $\beta \in \mathbb{R}$ such that $u' = \alpha u + \beta$ and $I'(\alpha a + \beta) = \alpha I(a) + \beta$ for any $a \in B(\Sigma,u(X))$.*

Next, we show that Axiom 6, *Dominance*, holds if and only if a linear functional J can be defined. This identifies a reference prior. It is shown that I coincides with J on all complementary acts.

Suppose that $\succsim$ is represented by a pair (u, I). Define $J : u \circ \mathcal{F} \to \mathbb{R}$ for $a \in u \circ \mathcal{F}$ and $\gamma \in \mathbb{R}$ with $\gamma - a \in u \circ \mathcal{F}$ as follows:

$$J(a) = \frac{1}{2}\gamma + \frac{1}{2}I(a) - \frac{1}{2}I(\gamma - a). \tag{2}$$

Lemma 2: *J is a well-defined, normalized neveloid.*[2] *If $\succsim$ satisfies Dominance, then J is affine on $\mathcal{F}$ and has a unique, normalized, and linear extension to $B(\Sigma)$.*

[2] See Maccheroni et al. (2006) in detail.

We can show that J is well-defined, and J is a normalize neveloid straightfoward. We omit the proof. Assume that $\succsim$ satisfies *Dominance*. First, we want to show that

$$J(\frac{1}{2}a) = \frac{1}{2}J(a) \quad . \tag{3}$$

Consider two complementary pairs ($f, \overline{f}$) and ($f', \overline{f'}$) such that $f \sim \overline{f}$ and $f' \sim \overline{f'}$. Without loss of generality, assume $\overline{f}$ and $\overline{f'}$ are constant acts. By *Dominance*, we can obtain

$$\frac{1}{2}f + \frac{1}{2}\overline{f'} \sim \frac{1}{2}f' + \frac{1}{2}\overline{f} \quad . \tag{4}$$

Then, by using the properties of $\overline{f}$ and $\overline{f'}$, we have the following:

$$I\left(\frac{1}{2}a\right) + \frac{1}{2}I(\gamma - a) = I\left(\frac{1}{2}(\gamma - a)\right) + \frac{1}{2}I(a) \quad . \tag{5}$$

By using the definition of J, it can be shown that $J(\frac{1}{2}a) = \frac{1}{2}J(a)$. By using this result and *Dominance*, we can show that

$$J\left(\frac{1}{2}a + \frac{1}{2}b\right) = \frac{1}{2}J(a) + \frac{1}{2}J(b) \quad . \tag{6}$$

Finally, we characterize the representation. We show that there exists $\theta \in [0,1]$ such that an act $f \in \mathcal{F}$ is evaluated by

$$\theta \sum_{s \in S} u(f(s))q(s) + (1-\theta)\sum_{s \in S}\left[u(f(s)) - \sum_{s \in S} u(f(s))q(s)\right] \quad . \tag{7}$$

Consider $f, g \in \mathcal{F}$ such that $f \sim g$. Before defining θ, let us introduce some notation. Let v be a function such that $v : X \times X \to \mathbb{R}$. Define

$$v(f, \overline{f}) = u \circ f - E_q(u \circ \overline{f}). \tag{8}$$

Define

$$\theta = \frac{\left|v(f,\overline{f}) - v(g,\overline{g})\right|}{[E_q(u \circ f) - v(f,\overline{f})] - [E_q(u \circ g) - v(g,\overline{g})]} \quad . \tag{9}$$

By *Certainty Independence*, it can be shown that θ does not depend on f and g.

3. Discussion

3.1. Machina(2009)'s Reflection Example

Table 1: Machina(2009)'s Reflection Example

	s_1	s_2	s_3	s_4
f_1	\$4000	\$8000	\$4000	\$0
f_2	\$4000	\$4000	\$8000	\$0
f_3	\$0	\$8000	\$4000	\$4000
f_4	\$0	\$4000	\$8000	\$4000

Machina(2009) gives the following situation. See Table 1. Suppose $S = \{s_1, s_2, s_3, s_4\}$. Assume that $\{s_1,s_2\}$ and $\{s_3,s_4\}$ are known to be equally likely. However, the relative likelihood between s_1 and s_2 and the relative likelihood between s_3 and s_4 are not known.

f_1 and f_4 are different from states where decision makers obtain the prizes. Similarly, f_2 and f_3 have the same structure. Then, it is easily to predict that decision makers have preferences such as $f_1 \sim f_4$ and $f_2 \sim f_3$. However, as Machina(2009) conjectured, L'Haridon and Placido(2009) experimentally studied that subjects preferred f_2 to f_1 and f_3 to f_4.

We provide a possible explanation of the above human behaviors by using our model. Since decision makers do not know the relative likelihood of $\{s_1,s_2\}$ and $\{s_3,s_4\}$, they can have a subjective belief as a reference prior $q = (q_1,q_2,q_3,q_4)$. Assume that $q = p_1 + p_2 = 1$ where $p_1 = q_1 + q_2$ and $p_2 = q_3 + q_4$. Notice that they know s_1 and s_2 are equally likely, and also that they know s_3 and s_4 are equally likely. If decision makers have $p_1 = p_2$,then the expected value with the reference prior q is \$4,000, so $f_1 \sim f_4$ and $f_2 \sim f_3$. If $p_1 \neq p_2$, then, by keeping $f_1 \sim f_4$ and $f_2 \sim f_3$, the decision maker identifies θ. With θ, the decision maker prefers f_2 to f_1 and f_3 to f_4. We give a numerical example. Let $p_1 = 3/5$ and $p_2 = 2/5$.[3] Then, we obtain $\theta = 4/5$. We obtain $v(f_2) > v(f_1)$ and $v(f_3) > v(f_4)$.

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[3] We can have $\theta = 4/5$ irrespective of the value of p_1 and p_2 if $p_1 \neq p_2$.