Precautionary Saving and Ambiguity

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Abstract

This paper considers a two period consumption-saving model in which future income is uncertain. If the future income is also ambiguous, in the sense of having multiple priors, then ambiguity attitudes also affect the saving decision. Unlike the static portfolio problem, ambiguity attitude does more than just distort the probabilities of the various priors. It also distorts the relative importance of second-period consumption, which in turn affects precautionary demand for saving. These effects can either reinforce or counteract the well known effects based upon risk attitudes in expected utility models.

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1 Introduction

The role of "prudence" in expected-utility models of consumption and saving has been known ever since the seminal works of Leland (1968), Sandmo (1970) and Dreze and Modigliani (1972). Kimball (1990) formalized the concept of prudence. Essentially, a convex marginal utility ("prudence") increases the benefit of saving when future income is risky. This extra saving is referred to as "precautionary saving."[1]

In this paper, we explore what might happen in a two-date consumption-saving model when there are several competing prior distributions for future income. In other words, the distribution of future income is *ambiguous*. Under expected utility, individuals are ambiguity neutral and simply aggregate the probabilities. But if individuals are ambiguity averse, how will it affect saving decisions?

Using the smooth ambiguity aversion models of Klibanoff, Marinacci and Mukerji (2005, 2009) – hereafter KMM (2005) and KMM (2009) respectively – and Neilson (2010), we show how ambiguity can affect saving decisions. KMM (2009) embeds ambiguity into a model of dynamic decision making. However, each stage of the process is a two-date static

[1]See Kimball (1992) for an introduction to this topic.

optimization. Thus, our focus on a two-date model of consumption and saving has a structure similar to this dynamic framework.

Essentially, we show that ambiguity aversion distorts one's preferences for consumption timing ("impatience") as well as the relative weights given to the competing prior distributions. The importance of consumption timing is similar to the discussion in Strzalecki (2013). This effect can also be found in the two-date self protection model of Berger (2011b). The distortion of probability weights for competing priors is analyzed within a one-period framework by Gollier (2011). As will be shown, the distortion is more complicated to interpret in our consumption-saving framework, since our distorted weights need not sum to one; i.e. the distortion embodies more than just a shift in probabilities.

For an expected-utility maximizer with utility u and with zero prudence ($u''' = 0$), there is zero precautionary demand for saving. Such is not always the case in the presence of ambiguity aversion. Indeed, even with zero risk prudence ($u''' = 0$), we show that precautionary saving can be either positive or negative under ambiguity aversion. Although Kimball (1990) showed that convex marginal utility, $u''' > 0$, guarantees a precautionary demand for saving under expected utility, the same does not hold under ambiguity aversion. Again, negative precautionary saving is also a possibility. This possibility exists even if the second order utility of Klibanoff et al. (2005), ϕ, has a marginal utility that is convex ($\phi''' > 0$). In other words, both $u''' > 0$ and $\phi''' > 0$ are not sufficient to guarantee a precautionary demand for saving.

In each of the above settings, what matters about ambiguity aversion is how its local intensity changes in response to increases in expected utility, where the local measure of absolute ambiguity aversion, $-\phi''/\phi'$, is as defined in KMM (2005). For instance, when $-\phi''/\phi'$ is an increasing function, future uncertainty becomes less important. This by itself would tend to decrease saving, so that it is possible for an ambiguity-averse individual with both $u''' > 0$ and $\phi'' > 0$ to actually have a negative demand for precautionary saving.

The above complications make it difficult to compare saving behaviors of different individuals. Given the same underlying (first-order) utility function, a more ambiguity-averse individual facing the same consumption-saving decision will not always save more. However, when the measure of absolute ambiguity aversion is constant, we show that a more ambiguity averse individual will indeed increase her level of precautionary saving.

The following section introduces our basic model of consumption and saving. We next examine an application of ambiguity without using the dynamic framework for ambiguity aversion; after all, our model is not a dynamic one. We show how this model has some undesirable properties. We then allow for the certainty equivalence proposed in KMM (2009) and establish conditions under which precautionary saving will exist, as well as conditions under which precautionary saving will be more than (or less than) would be the case absent any ambiguity aversion. Finally, we examine conditions under which comparative ambiguity aversion gives us definitive qualitative results.

2 Precautionary Saving and Expected Utility

There are two dates indexed $t = 0$ and $t = 1$. An individual earns sure income w at date $t = 0$, but faces a random income $\widetilde{w}_1$ at date $t = 1$. At date $t = 0$, the individual decides how much of her wealth w to consume. The rest is saved at a risk-free rate of interest. To keep the exposition simple, we assume that $\widetilde{w}_1 = w + \widetilde{\epsilon}$, where $\widetilde{\epsilon}$ is a zero-mean random income shock. We also assume that the risk-free rate of interest is zero and that the individual is not impatient, in the sense that future utility need not be discounted. All of these assumptions can be relaxed, but add complicated nuances to the basic model.

In an expected utility (EU) setting, the individual wants to choose a savings level s to maximize lifetime expected utility

$$U(s) \equiv u(w - s) + Eu(w + \widetilde{\epsilon} + s), \tag{1}$$

where u is an increasing and strictly concave utility function. Concavity connotes both a preference for intertemporal consumption smoothing as well as intra-period risk aversion. If $\widetilde{\epsilon}$ is degenerate (with a zero variance), then optimal saving is zero. If $\widetilde{\epsilon}$ is not degenerate, then optimal saving is positive if the individual is prudent, i.e. if $u'''(w) > 0$, see for example Kimball (1990). If this inequality is reversed, the individual is said to be imprudent and she borrows at the risk-free rate (a negative saving). If utility is quadratic with $u''' = 0$, then the optimal saving remains at zero. Any extra saving for the case where $\widetilde{\epsilon}$ is non-degenerate is so-called "precautionary saving."

We now introduce ambiguity and suppose that there are $n \geq 2$ possible distributions for $\widetilde{\epsilon}$, each leading to a conditional random variable $\widetilde{\epsilon}_\theta$ for $\theta = 1, ..., n$, with distribution functions $F_\theta(\epsilon)$. We further assume that $E\widetilde{\epsilon}_\theta = 0$ for each θ and that each F_θ has a support contained in the interval (a, b). Over the space of potential distributions of $\widetilde{\theta}$, the individual chooses a subjective set of probabilities q_θ for the likelihood of $\widetilde{\epsilon}_\theta$ being the true random income. Under expected utility, we simply set $\widetilde{\epsilon} = \sum_\theta q_\theta \widetilde{\epsilon}_\theta$. As is well known, ambiguity has no effect on saving decisions in an EU setting.

3 Ambiguity Aversion without Certainty Equivalence

Following KMM (2005) and Neilson (2010), we define an increasing second-order utility ϕ over the (first-order) utility of wealth. The function ϕ is assumed to be thrice differentiable. For each wealth w and each $\widetilde{\epsilon}_\theta$, $\phi(Eu(w+\widetilde{\epsilon}_\theta))$ denotes the second order utility derived from the expected utility $Eu(w + \widetilde{\epsilon}_\theta)$. Under ambiguity, expected second-order utility is thus $\sum_\theta q_\theta \phi(Eu(w + \widetilde{\epsilon}_\theta))$. If ϕ is linear, ambiguity has no effect on underlying intra-period preferences, since ϕ only induces an affine transformation of utility u. If ϕ is strictly concave, then

$$\sum_\theta q_\theta \phi(Eu(w + \widetilde{\epsilon}_\theta)) < \phi(Eu(w + \sum_\theta q_\theta \widetilde{\epsilon}_\theta)) = \phi(Eu(w + \widetilde{\epsilon})), \tag{2}$$

indicating an aversion to the ambiguity.

More recently, the literature has also examined the certainty equivalent for second order utility: $\phi^{-1}[\sum_{\theta} q_{\theta}\phi(Eu(w+\widetilde{\epsilon}_{\theta}))]$. It is important to note that this particular "certainty equivalence" is with respect to first-order utility u and *not* with respect to consumption. With no ambiguity, we simply revert to first-order utility (EU). In a one-period optimization problem with ambiguity, such as the static portfolio choice model examined by Gollier (2011), certainty equivalence is not particularly relevant; since maximizing $\phi^{-1}[\sum_{\theta} q_{\theta}\phi(Eu(\widetilde{w}_{\theta}(\alpha)))]$, with an endogenous wealth $\widetilde{w}_{\theta}(\alpha)$, is equivalent to maximizing $\sum_{\theta} q_{\theta}\phi(Eu(\widetilde{w}_{\theta}(\alpha)))$. However, the same cannot be said for a multiperiod model, as was also noted by Berger (2011b) and Strzalecki (2013).

Consider the following example without certainty equivalence. Let $u''' = 0$, so that optimal saving is zero in our two-period model of precautionary saving under EU. Since ϕ does transform utility, we apply it in both periods to find s to maximize

$$\phi(u(w-s)) + \sum_{\theta} q_{\theta}\phi(Eu(w+\widetilde{\epsilon}_{\theta}+s)). \tag{3}$$

This yields a first-order condition[2]

$$-\phi'(u(w-s))u'(w-s) + \sum_{\theta} q_{\theta}\phi'(Eu(w+\widetilde{\epsilon}_{\theta}+s))Eu'(w+\widetilde{\epsilon}_{\theta}+s) = 0. \tag{4}$$

Since we assume $u''' = 0$, we have $Eu'(w+\widetilde{\epsilon}_{\theta}+s) = u'(w+s)\ \forall\theta$. It thus follows in a straightforward manner that if ϕ and u are both strictly concave, $s^* > 0$. Ambiguity

[2] The second order condition is trivial to verify.

aversion alone (even with $u''' = 0$) would seem to imply a precautionary motive for saving.

But consider the case where $q_1 = 1$, so there is no ambiguity. We still have $\phi'(u(w)) < \phi'(Eu(w + \widetilde{\epsilon}_1))$. Thus, with $u''' = 0$, it follows from (4) that the optimal level of saving is still positive, $s^* > 0$. This does not seem like a particularly nice result. Utility gives no grounds for a precautionary saving motive, since $u''' = 0$. Moreover, there is no ambiguity. Yet, we still get precautionary saving.

Obviously, we can explore more results in this setting, but using a certainty equivalence seem to provide a "cleaner" set of results, as we show in the next section.

4 Introducing Certainty Equivalence

Since we will want to examine properties of the second order utility ϕ, we will follow Baillon (2013) and refer to properties related to utility u as "risk" properties, such as "risk prudence." A model of ambiguity aversion with certainty equivalence was established by KMM (2009). Using this version of ambiguity preferences, we can rewrite the objective function (3) as

$$V(s) \equiv u(w - s) + \phi^{-1}[\textstyle\sum_\theta q_\theta \phi(Eu(w + \widetilde{\epsilon}_\theta + s))]. \tag{5}$$

Reconsidering the case where $u''' = 0$ and $q_1 = 1$, so that there is no ambiguity and zero risk prudence, it follows trivially that optimal saving remains at zero. In other words, with zero risk prudence and no ambiguity, there is no precautionary demand for saving.

In models without certainty equivalence, see for example Baillon (2013), $\phi''' > 0$ is sufficient to generate a precautionary saving, in the case where we also have risk prudence, $u''' > 0$.[3] To see that this result need not hold with certainty equivalence, we first consider the case where we have zero risk prudence, $u''' = 0$. Again in this case, we have that $Eu'(w + \widetilde{\epsilon}_\theta) = u'(w)\ \forall\theta$ and $\forall w$. We can thus evaluate $V'(s)$ at $s = 0$:

$$V'(0) \equiv -u'(w) + \frac{\sum_\theta q_\theta \phi'(Eu(w + \widetilde{\epsilon}_\theta))}{\phi'[\phi^{-1}[\sum_\theta q_\theta \phi(Eu(w + \widetilde{\epsilon}_\theta))]]} u'(w). \tag{6}$$

To evaluate (6), we implicitly define the *ambiguity premium* π_A as in Berger (2011a): $\sum_\theta q_\theta \phi(z(\theta)) \equiv \phi[\sum_\theta q_\theta z(\theta) - \pi_A]$, where $z(\theta)$ denotes (first order) expected utility when the true random income risk is $\widetilde{\epsilon}_\theta$. The ambiguity premium is positive whenever the individual is ambiguity averse, $\phi'' < 0$.[4] Here, we also wish to consider a precautionary premium, as was defined by Kimball (1990) for utility u. In particular, we implicitly define the *ambiguity precautionary premium* ψ_A via $\sum_\theta q_\theta \phi'(z(\theta)) \equiv \phi'[\sum_\theta q_\theta z(\theta) - \psi_A]$. The ambiguity precautionary premium ψ_A is easily seen to be positive if ϕ' is strictly convex. In such a case, we adopt the terminology of Baillon (2013) and say that preferences are *ambiguity prudent*. Similarly, ψ_A is easily seen to be zero if ϕ' is linear.

The expression $\phi^{-1}[\sum_\theta q_\theta \phi(z(\theta))]$ is the certainty equivalent of $z(\widetilde{\theta})$ under ϕ. Thus, we obtain $\phi^{-1}[\sum_\theta q_\theta \phi(z(\theta))] = \sum_\theta q_\theta z(\theta) - \pi_A$. In our application, we set $z(\theta) = Eu(w + \widetilde{\epsilon}_\theta)$.

[3] Baillon (2013) sets up a very general model of higher order ambiguity attitudes, without certainty equivalence. His focus is a static one-period framework, although he does consider an application to precautionary saving. In this framework, ambiguity aversion alone is enough to distort preference towards consumption smoothing – even absent any ambiguity.

[4] The ambiguity premium π_A as defined here is defined in units of first-order utility. It is not defined in terms of units of consumption.

Assuming that ϕ is strictly concave, it follows easily from Pratt (1964) that $\pi_A > 0$, see also Berger (2011a). It is easy to see that the denominator of the fraction in (6) is equal to $\phi'(\sum_\theta q_\theta z(\theta) - \pi_A)$, while the numerator is equal to $\phi'(\sum_\theta q_\theta z(\theta) - \psi_A)$. Since ϕ' is strictly decreasing, this fraction is greater than 1 iff $\psi_A > \pi_A$.

First note that if $\phi'' < 0$ and $\phi''' = 0$, then $\psi_A = 0 < \pi_A$. It thus follows from (6) that we must have a negative optimal level of saving, $s^* < 0$. By continuity of preferences, it follows that for a small enough increase in ϕ''', we will still obtain $s^* < 0$. In other words, $u''' = 0$ and $\phi''' > 0$ are *not sufficient* to guarantee precautionary saving.

Let $\lambda(z) \equiv -\phi''(z)/\phi(z)$ denote the coefficient of absolute ambiguity aversion, as defined by KMM (2005). Again, analogous to the risk premium and the risk precautionary premium, it is easy to show[5] that $\psi_A > \pi_A$ for all possible random $z(\widetilde{\theta})$ iff the coefficient of absolute ambiguity aversion $\lambda(z)$ is decreasing in z (*decreasing absolute ambiguity aversion or DAAA*). When $\lambda(z)$ is constant (*constant absolute ambiguity aversion* or *CAAA*), we obtain $\psi_A = \pi_A$ and when $\lambda(z)$ is increasing in z, we obtain $\psi_A < \pi_A$. Our earlier assumption that $\phi''' = 0$ is an example of this last case, with so-called *increasing absolute ambiguity aversion* (*IAAA*). We thus obtain the following result:

Proposition 1: *If $u''' = 0$, then*

(*i*) *precautionary saving is positive under DAAA*

(*ii*) *precautionary saving is zero under CAAA*

[5] See, for example, Gollier (2001), who shows the analogous result for EU.

(iii) *precautionary saving is negative under IAAA*

We wish to reiterate that $\phi''' > 0$ is not sufficient to guarantee precautionary saving. When $u''' = 0$, any precautionary effect stems strictly from the fraction in (6) being greater than one. Such a circumstance is equivalent to increasing the weight of utility received at date $t = 1$. It a certain sense, DAAA exacerbates the importance of any income risk at a later date. This is quite similar to the dynamic model of Strzalecki (2013) who labels this effect "a preference for the earlier resolution of uncertainty."[6]

It also follows from Proposition 1 (i) and the smoothness of preferences that even with slight risk imprudence (i.e. u''' is very slightly negative) we can obtain a positive precautionary demand for saving under DAAA. Although an EU maximizer would have a negative saving, the increased disutility of uncertainty later in life under DAAA might cause overall effect to be an increase in saving.

In a similar manner, we can have u''' slightly positive with IAAA and have s* negative. The EU maximizer would choose a positive level of saving, but the IAAA preferences mitigate the importance of the future income uncertainty. From Proposition 1 (iii) and the continuity of preferences, it follows that $u''' > 0$ together with $\phi''' > 0$ need not guarantee a positive level of saving.

[6]See Strzalecki (2013) Theorem 4 and set his rate of time preference $\beta = 1$. His "preference for an earlier resolution of uncertainty" can be restated as an increased dislike for uncertainty that is resolved in later periods.

5 Precautionary Saving under Ambiguity Aversion

We now turn to cases in which the individual is risk prudent, with $u''' > 0$. This complicates the model in that we no longer have $Eu'(w + \widetilde{\epsilon}_\theta)$ equal to the same constant for each θ. Indeed, the convexity of u' yields $Eu'(w + \widetilde{\epsilon}_\theta) > u'(w)$. The first order condition for (5) is now

$$V'(s) \equiv -u'(w-s) + \frac{\sum_\theta q_\theta \phi'(Eu(w + \widetilde{\epsilon}_\theta + s))}{\phi'[\phi^{-1}[\sum_\theta q_\theta \phi(Eu(w + \widetilde{\epsilon}_\theta + s))]]} Eu'(w + \widetilde{\epsilon}_\theta + s) = 0. \qquad (7)$$

In the previous section, we already showed that $\phi''' > 0$ together with $u''' > 0$ is not sufficient to guarantee a precautionary saving. In this section, we examine conditions under which ambiguity increases the level of precautionary saving. Since we assume risk prudence, $u''' > 0$, it follows that the EU maximizer will choose a positive level of saving. Let $s_0 > 0$ denote the solution to (1). If the individual is ambiguity neutral, $\phi'' = 0$, she would choose the same optimal level of saving s_0. For the ambiguity averse individual, the variation in $Eu'(w + \widetilde{\epsilon}_\theta + s_0)$ as θ changes requires us to make some other assumptions if we wish to obtain definitive effects of ambiguity aversion.

Analogous to Gollier (2011), we assume that the prior distributions for $\widetilde{\epsilon}$ can be ranked by second order stochastic dominance. Since the risk $\widetilde{\epsilon}_\theta$ is assumed to have a zero mean for each θ, this is a ranking by *mean-preserving increases in risk* as defined by Rothschild and Stiglitz (1970). In particular, we assume that $\widetilde{\epsilon}_{\theta+1}$ is riskier than $\widetilde{\epsilon}_\theta$ in the sense of Rothschild and Stiglitz for $\theta = 1, ..., n-1$.

To see whether the solution s^* to (7) is greater than s_0, we evaluate $V'(s)$ when $s = s_0$. For the EU maximizer we know that

$$-u'(w - s_0) + \sum_\theta q_\theta Eu'(w + \tilde{\epsilon}_\theta + s_0) = 0. \tag{8}$$

For our ambiguity averter, we have

$$\begin{aligned} V'(s_0) = -u'(w - s_0) + \frac{\sum_\theta q_\theta \phi'(Eu(w+\tilde{\epsilon}_\theta+s_0))}{\phi'[\phi^{-1}[\sum_\theta q_\theta \phi(Eu(w+\tilde{\epsilon}_\theta+s_0))]]} \sum_\theta q_\theta Eu'(w + \tilde{\epsilon}_\theta + s_0) \\ + \frac{cov(\phi'(Eu(w+\tilde{\epsilon}_\theta+s_0)), Eu'(w+\tilde{\epsilon}_\theta+s_0))}{\phi'[\phi^{-1}[\sum_\theta q_\theta \phi(Eu(w+\tilde{\epsilon}_\theta+s_0))]]}. \end{aligned} \tag{9}$$

The term $Eu(w + \tilde{\epsilon}_\theta + s_0)$ is decreasing in θ and the term $Eu'(w + \tilde{\epsilon}_\theta + s_0)$ is increasing in θ, since $u'' < 0$ and $u''' > 0$ (see Rothschild and Stiglitz (1970)). Since ϕ' is a strictly decreasing function, it follows that the covariance term in (9) must be strictly positive.

As argued in demonstrating Proposition 1, the term $\sum_\theta q_\theta \phi'(Eu(w+\tilde{\epsilon}_\theta+s_0))$ is greater than $\phi'[\phi^{-1}[\sum_\theta q_\theta \phi(Eu(w + \tilde{\epsilon}_\theta + s_0))]]$ iff $\psi_A > \pi_A$ for the risk $\tilde{z}(\theta) \equiv Eu(w + \tilde{\epsilon}_\theta + s_0))$. This holds in the case of DAAA. Indeed, one can see this readily by re-writing (9) as follows:

$$\begin{aligned} V'(s_0) = -u'(w - s_0) + \frac{\phi'(\Sigma_\theta q_\theta(Eu(w+\tilde{\epsilon}_\theta+s_0)) - \psi_A)}{\phi'(\Sigma_\theta q_\theta(Eu(w+\tilde{\epsilon}_\theta+s_0)) - \pi_A)} \sum_\theta q_\theta Eu'(w + \tilde{\epsilon}_\theta + s_0) \\ + \frac{cov(\phi'(Eu(w+\tilde{\epsilon}_\theta+s_0)), Eu'(w+\tilde{\epsilon}_\theta+s_0))}{\phi'[\phi^{-1}[\sum_\theta q_\theta \phi(Eu(w+\tilde{\epsilon}_\theta+s_0))]]}. \end{aligned} \tag{10}$$

Comparing (10) to (8), it follows in this case that $V'(s_0) > 0$, which in turn implies that $s^* > s_0$.

If we have CAAA, then $\psi_A = \pi_A$ and we can use arguments similar to those above to see once again that $s^*> s_0$. If however we have IAAA, then $\psi_A < \pi_A$. Since the covariance term is positive, comparing (10) to (8) does not yield an unambiguous comparison between s^* and s_0.

These results are summarized as follows:

Proposition 2: *If $u''' > 0$ and if the $\widetilde{\epsilon}_\theta$ risks can be ranked via second order stochastic dominance then precautionary saving*

(i) will increase in the presence of ambiguity under DAAA or CAAA

(ii) may either increase or decrease in the presence of ambiguity under IAAA.

When $u''' > 0$ and the $\widetilde{\epsilon}_\theta$ are ranked via second order stochastic dominance, the positive covariance effect reflects a higher relative weighting on $Eu'(w + \widetilde{\epsilon}_\theta + s_0)$ for higher values of θ, i.e. for states with more $\widetilde{\epsilon}_\theta$ risk. But this effect needs to be weighed against the effects of changes in the importance of the timing of the uncertainty.

To isolate this "timing of uncertainty effect," define $\beta(s) \equiv \frac{\sum_\theta q_\theta \phi'(Eu(w+\widetilde{\epsilon}_\theta+s))}{\phi'[\phi^{-1}[\sum_\theta q_\theta \phi(Eu(w+\widetilde{\epsilon}_\theta+s))]]}$. When $u''' = 0$, the covariance term in (9) is also zero. Thus, we obtain $V'(s_0) = -u'(w - s_0) + \beta(s_0) \sum_\theta q_\theta Eu'(w + \widetilde{\epsilon}_\theta + s_0)$. Under CAAA, $\beta(s) = 1 \; \forall s$. Hence, the no ambiguity level of saving s_0 remains optimal, as in Proposition 1. When $u''' > 0$, ambiguity amplifies Eu' for the riskier distributions of $\widetilde{\epsilon}_\theta$, our "covariance effect," which leads to increased saving.

Under DAAA, $\beta(s) > 1\ \forall s$. When $u''' = 0$, this has the opposite effect as would "impatience" in consumption. Consumption at date $t = 1$ becomes more important than at date $t = 0$, leading to increased saving. When $u''' > 0$, ambiguity also re-weights the Eu' terms. This effect reinforces the effect of a lower "impatience."

Under IAAA, $\beta(s) < 1\ \forall s$. Thus, when $u''' > 0$, one effect of ambiguity is to increase impatience, and hence reduce saving; but a second effect re-weights the Eu' terms in such a way as to induce more saving. Thus, we cannot determine *a priori* whether saving will be higher or lower under ambiguity for this case. When u''' is only very slightly positive, we might still obtain negative saving, which is less than the positive saving of an EU maximizer with $u''' > 0$. But we can also have increased saving or no change in saving under ambiguity with IAAA. Consider the following example.

Example: Let $u(w) \equiv -e^{-aw}$. Thus u exhibits CARA and is prudent. Let $\phi(z) \equiv -(-z)^2$. Since z is negative here, it follows that $\phi' > 0$, $\phi'' < 0$ and $\phi''' = 0$. It also follows that $\lambda(z) \equiv -\phi''(z)/\phi(z) = (-z)^{-1}$, which exhibits IAAA. Also, it is easy to solve for $\phi^{-1}(\phi) = -(-\phi)^{1/2}$, where ϕ is of course negative here. Some calculation shows that

$$\phi^{-1}[\textstyle\sum_\theta q_\theta \phi(Eu(w + \widetilde{\epsilon}_\theta + s_0))]] = -[\textstyle\sum_\theta q_\theta (E\exp(-a(w + \widetilde{\epsilon}_\theta + s_0))^2]^{\frac{1}{2}} \tag{11}$$

Differentiating with respect to s and comparing $V'(s_0)$ in (9) and (8), it follows that optimal saving will increase if

$$\textstyle\sum_\theta q_\theta E(\exp(-a(w+\widetilde{\epsilon}_\theta+s_0))^2-[\sum_\theta q_\theta E(\exp(-a(w+\widetilde{\epsilon}_\theta+s_0))]^2>0. \qquad (12)$$

But this inequality always holds, since the left-hand side of inequality (12) is simply the variance of $x(\widetilde{\theta})$, where $x(\theta)\equiv E\exp(-a(w+\widetilde{\epsilon}_\theta+s_0)$. Thus, we have s*$> s_0$, even though we have IAAA with $u'''>0$.

We can extend the results in Proposition 2 to cases where the $\widetilde{\epsilon}_\theta$ can by ranked via N^{th} order stochastic dominance, for any $N\geq 2$.[7] In such a case, we need to add the assumption that $\text{sgn}(u^n)=(-1)^{n+1}$ for $n=2,...,N+1$. The term $Eu(w+\widetilde{\epsilon}_\theta+s_0)$ is then decreasing in θ and the term $Eu'(w+\widetilde{\epsilon}_\theta+s_0)$ is increasing in θ, so that the covariance term in (9) remains strictly positive. See, for example, Eeckhoudt and Schlesinger (2008). Thus, we obtain the following extension of Proposition 2:

Corollary 1: *If* $\text{sgn}(u^n)=(-1)^{n+1}$ for $n=2,...,N+1$ *and if the* $\widetilde{\epsilon}_\theta$ *risks can be ranked via* N^{th} *order stochastic dominance, then precautionary saving*

(i) will increase in the presence of ambiguity under DAAA or CAAA

(ii) may either increase or decrease in the presence of ambiguity under IAAA.

[7] Such an extension can also be made to an "increase in nth degree risk" as defined by Ekern (1980).

6 Comparative Ambiguity Aversion

From Proposition 1, which compares ambiguity neutrality to ambiguity aversion, it follows that neither higher ambiguity aversion nor higher ambiguity prudence will necessarily lead to more saving. This follows since we cannot compare them when $u''' = 0$. One person may have CAAA and the other DAAA or IAAA for example. Using continuity arguments, we also cannot compare individuals if u''' is slightly positive!

Unlike in Gollier (2011), our $\beta(s)$ as defined in the previous section need not equal 1. Therefore, our first order condition (7) is not a simple distortion of probability weights, as explained previously. Only for the special case where have CAAA do we obtain $\beta(s) = 1$. We examine here whether or not a more ambiguity averse individual will have more precautionary saving under an assumption of CAAA. Of course, if $u''' = 0$, we obtain zero precautionary saving. We assume here that the individual is risk prudent with $u''' > 0$.

Consider two risk-prudent individuals with the same underlying utility function u. Both individuals are ambiguity averse with second order utilities $\phi_1(z)$ and $\phi_2(z)$ respectively. We assume that individual 2 is more ambiguity averse than individual 1. Thus, we assume constants $\lambda_2 > \lambda_1$ such that $\frac{-\phi_2''(z)}{\phi_2(z)} = \lambda_2$ and $\frac{-\phi_1''(z)}{\phi_1'(z)} = \lambda_1 \ \forall z$. Let s_1 denote the optimal saving for individual 1. This level of saving is positive, $s_1 > 0$, by Proposition 2. We now wish to determine whether the optimal saving for individual 2, s_2, is higher than s_1.

In the special case of CAAA, the ambiguity premium and the ambiguity precautionary premium are equal to each other. It follows that, for $i = 1, 2$,

$$\phi_i'[\phi_i^{-1}[\textstyle\sum_\theta q_\theta \phi_i(Eu(w + \widetilde{\epsilon}_t + s))]] = \textstyle\sum_\theta q_\theta \phi_i'(Eu(w + \widetilde{\epsilon}_\theta + s)). \tag{13}$$

This allows us to rewrite $V_i'(s_1)$ from (7) as follows:

$$V_i'(s_1) = -u'(w - s_1) + \textstyle\sum_\theta \widehat{q}_{i\theta} Eu'(w + \widetilde{\epsilon}_\theta + s_1), \tag{14}$$

where

$$\widehat{q}_{i\theta} = q_\theta \frac{\phi_i'(Eu(w + \widetilde{\epsilon}_\theta + s_1))}{\sum_t q_t \phi_i'(Eu(w + \widetilde{\epsilon}_t + s_1))}. \tag{15}$$

Note that $\widehat{q}_{i\theta}$ is now a type of ambiguity-neutral probability, similar to that in Gollier (2011). In other words, individual i acts like an EU maximizer with transformed probabilities for each $\widetilde{\epsilon}_\theta$. Of course (14) is zero for individual 1, since s_1 is optimal.

Since $\phi_2(z)$ is more ambiguity averse than $\phi_1(z)$, there exists a strictly increasing and concave function h such that $\phi_2(z) = h(\phi_1(z))\ \forall z$, as shown by KMM (2005). Hence, we can write $\phi_2'(z) = h'(\phi_1(z))\phi_1'(z)$. For every θ, we thus have

$$\frac{\widehat{q}_{2\theta}}{\widehat{q}_{1\theta}} = h'(\phi_1(Eu(w + \widetilde{\epsilon}_\theta + s_1))) \frac{\sum_t q_t \phi_1'(Eu(w + \widetilde{\epsilon}_t + s_1))}{\sum_t q_t \phi_2'(Eu(w + \widetilde{\epsilon}_t + s_1))}. \tag{16}$$

Since $Eu(w + \widetilde{\epsilon}_\theta + s_1)$ is decreasing in θ, and since the composite function $h' \circ \phi_1$ is decreasing, it follows that the ratio $\widehat{q}_{2\theta}/\widehat{q}_{1\theta}$ is decreasing in θ. Thus, the ambiguity-neutral probabilities $\widehat{q}_{2\theta}$ are dominated by the ambiguity-neutral probabilities $\widehat{q}_{1\theta}$ via the monotone likelihood ratio property.

Let $\widetilde{\epsilon}(i) \equiv \sum_{\theta} \widehat{q}_{i\theta}\widetilde{\epsilon}_{\theta}$. Note that we retain the property that $E\widetilde{\epsilon}(i) = 0$ for both $i = 1$ and $i = 2$. It follows from Gollier (2011, Proposition 1) that $\widetilde{\epsilon}(1)$ dominates $\widetilde{\epsilon}(2)$ via second order stochastic dominance. As a result, since $u''' > 0$, it follows that $V_2'(s_1) > V_1'(s_1) = 0$.[8] This is establishes the following result:

Proposition 3: *Suppose $u''' > 0$ and the $\widetilde{\epsilon}_{\theta}$ risks can be ranked via second order stochastic dominance. Further assume that ambiguity preferences satisfy CAAA. Than an individual who is more ambiguity averse in the sense of KMM (2005) will choose a higher level of precautionary saving.*

We can also easily extend Proposition 3 to cases where the $\widetilde{\epsilon}_{\theta}$ can be ranked by N^{th} order stochastic dominance. It is straightforward in this case to extend Proposition 1 in Gollier (2011), who considers only first and second order stochastic dominance, to stochastic dominance of any order N.[9] This leads to the following result.

Corollary 2: *Suppose that ambiguity aversion satisfies CAAA and that individual 2 is more ambiguity averse than individual 1. Further suppose that* $\mathrm{sgn}(u^n) = (-1)^{n+1}$ for $n = 2, ..., N+1$ *and that the $\widetilde{\epsilon}_{\theta}$ risks can be ranked via N^{th} order stochastic dominance,*

[8] This result follows from Eeckhoudt and Schlesinger (2008), who consider changes in the distribution of income risk under EU.

[9] This follows since an expected utility maximizer with $\mathrm{sgn}(u^n) = (-1)^{n+1}$ for $n = 2, ..., N+1$ will rank the $\widetilde{\epsilon}_{\theta}$ via the stochastic dominance. We can then use the fact that the re-weighted relative probabilities $\widehat{q}_{2\theta}/\widehat{q}_{1\theta}$ satisfy the monotone likelihood ratio property.

Then precautionary saving is higher for individual 2 than for individual 1.

7 Concluding Remarks

We considered a simple model of precautionary saving when shocks to future income are ambiguous. A model without certainty equivalence was shown to have undesirable properties. Such preferences can alter saving decisions even when there is no ambiguity present.

For a model with certainty equivalence, there are two effects that must be compared when measuring the consequences of ambiguity aversion. One effect is a shift in the relative weighting of the various prior distributions of future income. Similar to Gollier's (2011) model of portfolio choice, an ambiguity averse individual shifts more relative weight to "worse" prior distributions. But a second effect re-weights the overall importance of second period consumption and hence the importance of the second-period uncertainty. This effect is analogous to the "preference for the earlier resolution of uncertainty" (under DAAA) found in the dynamic framework of Strzalecki (2013).

For an expected utility maximizer, risk prudence ($u''' > 0$) is known to generate a precautionary demand for saving. However, an ambiguity averse individual who is also risk prudent ($\phi''' > 0$) need not have a positive precautionary saving demand. Indeed, perhaps somewhat surprisingly, both $u''' > 0$ and $\phi''' > 0$ together are not sufficient for a precautionary demand.

Only in the case of constant absolute ambiguity aversion, do we not get this second effect of re-weighting the importance of second-period consumption. Under CAAA, if the prior distributions of future income can be ranked via second order stochastic dominance, a more ambiguity averse individual will have a higher level of precautionary saving.

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